CHAPTER 8

DEDUCIBILITY, ENTAILMENT AND ANALYTIC CONTAINMENT

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The concept of entailment is often connected with deducibility: A is said to entail B iff B is logically deducible from A.[1] It has also been connected to the concept of containment in Kant's sense of analytic containment: A entails B only if the meaning of B is contained in the meaning of A. But the concepts of deducibility and containment are two distinct concepts, and the failure to distinguish them leads to faulty attempts to merge them in formal systems. One such attempt is Anderson and Belnap's system, E, in which a Fitch-type theory of natural deduction is modified to incorporate a certain sense of "containment".[2] Another is Parry's system, AI, of "analytic implication" which began with a more restricted sense of containment but has usually been presented as a theory of deducibility (cf. Parry 33 and 72).

In this paper I first consider several effective criteria or conditions which are plausibly related to the containment, or sameness, of meanings in expressions. Secondly, I present a formal system, AC, which is shown to meet these conditions, treating entailment as analytic containment only, distinct from deducibility. Thirdly, concentrating on "tautological" first-degree entailments (i.e., entailments only between those sentences which are instances of truth-functional schemata), I relate this stronger concept of entailment to results in the systems E and AI. All three systems agree in rejecting the "paradoxes of strict implication", $(A \rightarrow (B \vee -B))$, $(A \,\&\, -A \rightarrow B)$, etc., on the ground that they express neither relations of containment nor of deducibility. But there are differences in the ways in which Anderson and Belnap on the one hand, and Parry on the other, compromise the concept of deducibility to accommodate a concept of containment, or vice versa. I conclude with some tentative suggestions on deducibility, hoping to have shown something of the utility gained by a clear-cut formalization of the stronger, less ambiguous, concept of analytic containment.

I

Turning to entailment as containment, I want to be faithful in an effective way to the dictum that S_1 entails S_2 only if the meaning of S_2 is contained in the meaning of S_1. Entailment in this sense is connected to synonymity: S_1 is synonymous with S_2 if and only if S_1 entails S_2 and S_2 entails S_1. Taken together these dicta yield the familiar proposition that S_1 is synonymous with S_2 if and only if they contain all and only the same meanings. The problem is to find an effective and plausible formalization which can represent containment of meanings; this will occupy the centre of attention. It is helpful to begin with the following criterion of adequacy for any proposed theory of entailment in the sense of containment of meanings:

I. A theory of entailment (as containment) is satisfactory only if: for all sentences S_1

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and S_2, if S_1 entails S_2 and S_2 entails S_1 according to this theory, then S_1 and S_2 contain all and only the same meanings.

Let 'A', 'B', 'C', be metalogical variables taking standard truth-functional schemata as values; i.e., they stand for any formulae built up by usual rules for formation and definitions from sentential variables, 'S_1', 'S_2', 'S_3', ..., parentheses, and the logical constants '&', '-', 'V', '⊃', '≡'. And let '(A → B)' represent the claim that A entails B. The schemata A and B in a theorem ⌜(A → B)⌝ of my theory, will be such that the meaning of B is contained in the meaning of A. The question is, how can one determine, by reference to syntactically determinate properties of the schemata A and B, that this relationship between meanings holds or fails?

A point of departure is found in the concept of variable-sharing, a concept used in different ways by both Parry and Anderson and Belnap (both are discussed below). If two schemata have the same variables, then by joint substitution they will have occurrences of the same sentences. It follows, by the principle that the same sentence has the same meaning in all of its occurrences, that joint instances of two schemata containing a common sentential variable will "contain" similar meanings in some parts at least. But this is much too loose. The "containment of meanings" related to entailment is much more restricted than "containment" in the sense of "occurrence in". One could claim that the meaning of S_3 must have an *occurrence* in any sentence ⌜(S_1 V S_3)⌝ or ⌜-(S_1 & S_3)⌝; but we surely do not want to say that ⌜(S_1 V S_3)⌝ would entail S_3, or that ⌜-(S_1 & S_3)⌝ entails S_3. Ordinarily, if S_1 entails S_2, we say that if S_1 were true S_2 would have to be true also. But it is not the case that if ⌜(S_1 V S_3)⌝ were true, then S_3 would have to be true also (even though S_3 occurs in ⌜(S_1 V S_3)⌝); so we deny that ⌜(S_1 V S_3)⌝ entails, or contains in the logical sense, S_3 in such cases. Since we are dealing only with truth-functional schemata this restriction on containment amounts to laying down the necessary conditions:

I<u>a</u>. If ⌜(A → B)⌝ is a theorem, then ⌜(A ⊃ B)⌝ is a theorem of standard logic; if ⌜(A ↔ B)⌝ is a theorem, then ⌜(A ≡ B)⌝ is a theorem of standard logic.

For on standard interpretations ⌜(A ⊃ B)⌝ is a theorem of logic only if B must be true if A is true, and ⌜(A ≡ B)⌝ is a theorem only if B is true if and only if A is. Where classical logicians went wrong was in the occasional suggestions that these were also sufficient conditions; that e.g., if ⌜(A ≡ B)⌝ is a theorem then all instances of A and B must "have the same meanings".

But now how about variable-sharing, *provided* condition I<u>a</u> is met, as a formal counterpart of entailment? There is a weaker and a stronger version of this criterion. The weaker version is:

I<u>b</u>. If ⌜(A → B)⌝ is a theorem, then B must contain at least one variable which occurs in A; if ⌜(A ↔ B)⌝ is a theorem, then A and B must share at least one variable.

This rules out the "paradoxes of strict implication", ⌜((A & -A) → B)⌝ and ⌜(A → (B ∨ -B))⌝ and related schemata which are theorem schemata of standard logic (with '⊃' for '→') and Lewis's modal logics (with '⥽' for '→'). Both Lewis (46, p.71) and Carnap (47, pp.60-61), in trying to deal with synonymity, agreed that truth-functional equivalence and strict equivalence were inadequate since they both make all inconsistencies equivalent and all logically true statements equivalent. It follows that neither truth-functional implication nor strict implication (i.e., theorems of the form ⌜(A ⊃ B)⌝ or ⌜(A ⥽ B)⌝) capture entailment in the sense of containing meanings. The stronger version of variable sharing as a necessary condition is:

> I<u>c</u>. If ⌜(A → B)⌝ is a theorem, then B contains only variables which occur in A; if ⌜(A ↔ B)⌝ is a theorem, then A and B contain all and only the same variables.

This condition eliminates all schemata eliminated by I<u>b</u> and in addition it eliminates ⌜(A → (A ∨ B))⌝ and ⌜(A ↔ (A & (A ∨ B)))⌝ (the so-called laws of Addition, and Absorption) among others. It leaves the principle of simplification, ⌜((A & B) → A)⌝ as a sort of paradigm of entailment. It is at this point that deducibility and containment begin to part company. For it seems clear that if S_1 is true, then ⌜(S_1 ∨ S_2)⌝ must be true as well, i.e., that we can *deduce* the truth of ⌜(S_1 ∨ S_2)⌝ from the truth of S_1. But it is not all that clear that the meaning of a sentence ⌜(S_1 ∨ S_2)⌝ is *contained in* the meaning of the sentence S_1. Similarly, it is clear that a sentence, S_1, will be true if and only if ⌜(S_1 & (S_1 ∨ S_2))⌝ is true; *the truth of* each is deducible from the *truth of* the other. But it also seems obvious that in general S_1 will not contain all and only the same meanings as ⌜(S_1 & (S_1 ∨ S_2))⌝. To admit the principle of Absorption as a principle of entailment in our present sense, would be to say that two sentences could contain all and only the same meanings even though one referred to and talked about individuals the others did not, and/or used predicates the other did not. Condition I<u>c</u> guarantees that S_2 will not have an occurrence of any simple predicate or singular term which does not occur in S_1. Even on a referential theory of meaning this seems necessary for a theory of entailment as containment of meanings. As we shall see, Anderson and Belnap's E satisfies conditions I<u>a</u> and I<u>b</u>, but not condition I<u>c</u>; Parry's analytic implication, on the other hand, satisfies all three conditions.

Two more even stronger syntactical conditions are required by the principle that two sentences can not have the same meanings if one says something false (or true, or inconsistent, or tautologous) about certain individual entities while the other does not.

Consider first schemata of the form ⌜((A & -A & B) ↔ (A & -B & B))⌝; such schemata satisfy conditions I<u>a</u>, I<u>b</u> and I<u>c</u>. But by the principle just mentioned we should not want to say that all of such schemata yielded true assertions of entailment in the sense of containment of meaning. For example, '(Jo died and Jo did not die and Flo wept)' does not mean the same as '(Jo died and Flo did not weep and Flo wept)'; for the first contains a false and inconsistent statement about Jo though the second does not, while the second contains a false and inconsistent statement about Flo though the first does not. How can two sentences mean the same thing if one contains a false and inconsistent statement about an individual while the other does not? A syntactical condition which will rule out such cases can be formulated by using a distinction by Herbrand[3] between 'positive' and 'negative' occurrences of a variable in a schema. Assuming (for simplicity) that we use just the primitive truth-functional connectives '-', and '&' or '∨', a variable *occurs negatively* in a purely truth-functional schema A, if and only if it lies in the scope of an odd number of negation signs in the primitive notation for A, and a variable *occurs positively* in a schema A, if and only if it does not occur negatively, i.e., if it occurs in the scope of zero or an even number of negation signs in the primitive notation of A. Then the following yields the required condition:

> I<u>d</u>. If ⌜(A → B)⌝ is a theorem, then each variable which occurs positively (negatively) in B, must occur positively (negatively) in A;
> If ⌜(A ↔ B)⌝ is a theorem, then a variable occurs positively (negatively) in B if and only if it occurs positively (negatively) in A.

This condition rules out ⌜((A & -A & B) ↔ (A & -B & B))⌝ which is a theorem in Parry's system, but not in Anderson and Belnap's, as well as others to be discussed shortly. But it is still not strong enough. For consider the schema ⌜(((A & -A) & (B ∨ -B)) ↔ ((A ∨ -A) & (B & -B)))⌝; this satisfies all of the conditions I<u>a</u>, I<u>b</u>, I<u>c</u> and I<u>d</u> but would still lead to violations of the principle mentioned above. Putting 'Jo died' for 'A' and 'Flo wept' for 'B' in this schema we get an assertion of mutual entailment or synonymity in which the left-hand expression asserts something inconsistent and false about Jo as well as something tautologous and true about Flo, while the right-hand expression asserts something true and tautologous about Jo and something inconsistent and false about Flo. Even if it were argued (speciously in my view) that inconsistencies and tautologies do not "assert" anything about anybody, the fact would remain that the same inconsistencies do not occur, and the same tautologies do not occur, in the two expressions. Thus on the view of Carnap and Lewis that different tautologies and different inconsistencies have different meanings, the two expressions will not mean the same thing or mutually entail each other. To give an effective syntactical condition which will rule out this example we define a certain very precise type of normal form (to be called a *maximal ordered normal form* of A). A tautology will be said to be "implicit" in A if it is a conjunct of the maximal ordered conjunctive normal form of A and an inconsistency will be said to be implicit in A if it is synonymous with a conjunction of 2^n different conjuncts of the basic conjunctive normal form of A each of which have the same set of n letters. Leaving the definition of maximal ordered conjunctive normal forms until later, the next condition can be formulated as follows:

> I<u>e</u>. If ⌜(A → B)⌝ is a theorem, then every tautology or inconsistency implicit in B must be implicit in A;
> If ⌜(A ↔ B)⌝ is a theorem, then a tautology or inconsistency is implicit in B if and only if it is implicit in A.

As we shall see this condition rules out the schema last considered, a schema which is a theorem in Parry's system but not in Anderson and Belnap's.

A final syntactical condition for systems of entailment in the sense of containment is expressed as follows:

I*f*. If ⌜(A → B)⌝ is a theorem, then every conjunct in the maximal ordered conjunctive normal form of B is a conjunct in the maximal ordered conjunctive normal form of A;
If ⌜(A ↔ B)⌝ is a theorem, A and B have identical maximal ordered normal forms.

It will follow from the definition of maximal ordered normal forms that if this condition is met then all of the preceding conditions, I*a* to I*e*, will be met as well, and I shall hold that this is not only a necessary but also a sufficient condition for theories of entailment in the sense of containment so far as truth-functional schemata are concerned. The intuitive or philosophical justification of this rule is not as easy to explain as was the case in the previous rules; although, if it be granted that the maximal ordered conjunctive normal form of a formula preserves sameness of meaning, then principles in I*f* amount to special cases of the principle of simplification which is connected in an obvious way with the concept of containment. Ultimately all justification must rest on a semantic theory of truth-conditions according to which two truth-functional sentential compounds will have the same meanings if and only if they have the same set of truth-conditions (not to be confused with "express the same truth-functions"). It must then be shown that instances of two truth-functional schemata will have the same sets of truth-conditions if and only if they have the same maximal ordered normal forms. The third system of entailment and synonymity to be presented satisfies this condition and thus all the others.

II

Now let us examine a system, AC (for 'analytic containment'), which will provably satisfy all of the criteria just laid down. Theorems of this system will be compared with appropriate sets of theorems from Anderson and Belnap's system E (for 'entailment') and Parry's system AI (for 'analytic implication') in the following section, with particular emphasis on points at which the latter systems go beyond the strict criteria for containment that we have laid down in the direction of different notions of deducibility. AC is formulated so that its theorems are confined to entailments (in the sense of containments) only between standard truth-functional schemata; i.e., to "first-degree entailments" or, when valid, "tautological entailments". Both of the systems, E and AI, contain higher than first-degree entailments, with occurrences of '→' lying in the scope of other occurrences of '→', but comparisons at this elementary level will be sufficient to establish most of the points relevant to our present purpose.

All three systems will be formalized with the same primitive symbols and rules of formation for truth-functional schemata, namely, '&' for "and", '-' for "not", parentheses for grouping devices, and 'S_1','S_2', ... as sentential variables, then, using 'A', 'B', 'C', 'D' as metalogical variables taking truth-functional wffs as values, well-formed schemata include all sentential variables, ⌜-A⌝ and ⌜(A & B)⌝, as well as what can be gotten from the usual definitions of '∨' (⌜(A ∨ B)⌝ =df ⌜-(-A & -B)⌝), '⊃' (⌜(A ⊃ B)⌝ =df ⌜(-A ∨ B)⌝) and '≡' (⌜(A ≡ B)⌝ =df ⌜((A ⊃ B) & (B ⊃ A))⌝). Further we have the symbols, '→' and '↔' which are the subject of discussion. In AI and E '→' is primitive and ⌜(A ↔ B)⌝ =df ⌜(A → B) & (B → A))⌝, while in AC '↔' is primitive and ⌜(A → B)⌝ =df ⌜(A ↔ (A & B))⌝ (although ⌜(A ↔ B)⌝ is derivable from ⌜(A → B)⌝ and ⌜(B → A)⌝ and vice-versa in AC).[4]

The system AC has the following axiom schemata and rules of inference:

AC1. (A ↔ --A)	Double Negation
AC2. (A ↔ (A & A))	Conjunctive Idempotence
AC3. ((A & B)↔ (B & A))	Conjunctive Commutation
AC4. ((A & (B & C)) ↔ ((A & B) & C))	Conjunctive Association
AC5. ((A ∨ (B & C)) ↔ ((A ∨ B) & (A ∨ C)))	Distribution
R1. From ⊢⌜(A ↔ B)⌝ and ⊢ X, infer ⊢X^A//B.	

We use 'X' and 'Y' for wffs, including those containing '↔', since 'A', 'B' and 'C' are reserved for truth-functional schemata only. The symbol 'X^A//B' means 'the result of replacing some occurrences of B in X by A'. We quickly obtain from AC the following theorem and derived rules:

AC6. (A ↔ A)

Proof:	1)	(A ↔ (A & A))	[AC2]
	2)	(A ↔ A)	[AC2,1,R1]

R2. (If ⊢ ⌜(A ↔ B)⌝ then ⊢ ⌜(B ↔ A)⌝

Proof:	1)	⊢ ⌜(A ↔ B)⌝	[Hyp]
	2)	⊢ ⌜(B ↔ B)⌝	[AC6]
	3)	⊢ ⌜(B ↔ A)⌝	[1,2,R1]

And by similar steps,

R3. If ⊢ ⌜(A ↔ B)⌝ then ⊢ ⌜(-A ↔ -B)⌝

R4. If ⊢ ⌜(A ↔ B)⌝ then ⊢ ⌜((A & C) ↔ (B & C))⌝

R5. If ⊢ ⌜(A ↔ B)⌝ and ⊢ ⌜(B ↔ C)⌝ then ⊢ ⌜(A ↔ C)⌝

The full systems of E and AI are presented and compared below; each of the axiom schemata AC1 - AC5 are derivable in these two systems as is the rule, R1, of the substitutivity of mutual entailments or analytic biconditionals. Thus biconditional theorems of AC are a sub-set of the biconditional theorems in both AI and E. The virtues of AC lie in the biconditionals it excludes as theorems, while AI or E includes them. By adding additional primitive rules of transformation (namely, "From ⊢⌜(A → B)⌝ infer ⊢ ⌜(A ⊃ B)⌝" and "From ⊢ A and ⊢ B infer ⊢ ⌜(A & B)⌝") we could add just the theorems of standard truth-functional logic to AC; and by other axioms or rules we could extend AC to a system with higher than first degree wffs as theorems. However, this is not needed for our present purposes. Such extensions would still be sub-systems of AI and E but the basic distinctions can be made at the simpler level.

In this section we show that AC meets all of the conditions, Ia-If, mentioned in Section I as conditions of entailment in the sense of containment. In the next section we show different ways in which AI and E meet, or fail to meet, these criteria for containment as well as some for deducibility. In the final section we venture a few remarks on distinctions between containment and deducibility.

That AC satisfies conditions Ia through Id can be established fairly simply:

Ia. If ⌜(A → B)⌝ is a theorem of AC, then ⌜(A ⊃ B)⌝ is a theorem of standard logic. If ⌜(A ↔ B)⌝ is a theorem of AC, then ⌜(A ≡ B)⌝ is a theorem of standard logic.

Proof: Replace '↔' and '→' throughout the system AC by '≡' and '⊃' respectively; all axiom schemata are then converted into truth-table tautologies, hence theorems of standard logic. Also the definition of ⌜(A → B)⌝ as ⌜(A ↔ (A & B))⌝ is admissible since ⌜((A ⊃ B) ≡ (A ≡ (A & B))⌝ is a truth-table tautology also. Since the rule of substitution, R1, is a derived rule for sentential logic, all derivable theorems will be theorems of standard logic. Hence for every proof of ⌜(A ↔ B)⌝ or ⌜(A → B)⌝ in AC, there is a corresponding proof for ⌜(A ≡ B)⌝ or ⌜(A ⊃ B)⌝ respectively, in standard logic.

(This proof also shows that AC is consistent, since it corresponds to a fragment of standard sentential logic which is consistent).

Since Ic implies Ib, we prove Ic first:

Ic. If ⌜(A → B)⌝ is a theorem of AC, then B contains only variables which occur in A; if ⌜(A ↔ B)⌝ is a theorem of AC, then A and B contain all and only the same variables.

Proof: Inspection of AC1 through AC5, all of which have the form ⌜(X ↔ Y)⌝, shows that in each of these axiom schemata X and Y contain all and only the same metavariables A,B, or C; thus all axioms gotten from these axiom schemata will have all and only the same set of sentential variables occurring on either side of '↔'. This property is preserved through the introduction and elimination of abbreviations, and by the use of the substitution rule laid down in R1.

In general, then, no theorem of the form ⌜(A ↔ B)⌝ will contain a sentential variable in A unless B contains it, or in B unless A contains it. Thus the second part of condition Ic holds of AC. But ⌜(A → B)⌝ is defined in AC as ⌜(A ↔ (A & B))⌝; thus ⌜(A → B)⌝ is a theorem of AC only if (A ↔ (A & B)) is. But in the latter case all sentential variables in B must be contained in A by our first result. Hence the first part of condition Ic holds in AC.

Ib. If ⌜(A → B)⌝ is a theorem of AC, then B contains at least one variable which occurs in A; if ⌜(A ↔ B)⌝ is a theorem of AC, then A and B have at least one variable in common.

Proof: Follows as a special case of Ic.

The proof that AC satisfies condition Id is similar to that for Ic, but slightly more complicated:

Id. If ⌜(A → B)⌝ is a theorem of AC, then each variable which occurs positively (negatively) in B occurs positively (negatively) in A; if ⌜(A ↔ B)⌝ is a theorem of AC, then a variable occurs positively (negatively) in B if and only if it occurs positively (negatively) in A.

Proof: Inspection of AC1 through AC5 shows that in each of these axiom schemata a metavariable A, B, or C, occurs in the scope of an odd number of negation signs (i.e., occurs negatively) on the left of '↔' if and only if it has an occurrence in the scope of an odd number of negation signs on the right of '↔'; since this test is applied only after reduction to primitive notation AC5 must first be reduced to '(-(-A & -(B & C)) ↔ (-(-A & -B) & -(-A & -C)))' where it is seen to apply - in AC1 through AC4 the application is obvious. Since the same schemata will replace all occurrences of the same metavariables to get axioms the second part of Id will hold of all axioms of AC gotten from AC1 through AC5. (Remember that abbreviations do not affect negative and positive occurrence properties since these properties are determined after reduction to primitive notation.) Further this property is preserved for all theorems gotten by substitutions based on the use of R1, or of derived rules R2 to R5, when applied to wffs which have the form ⌜(A ↔ B)⌝, so that the second part of Id holds in AC. But ⌜(A → B)⌝ is defined as ⌜(A ↔ (A & B))⌝ and from this (as in the proof for Ic) it follows that the first part of Id must hold in AC as well.

The proof of condition Ie will follow from the proof that condition If is met in AC. The latter proof is too long and detailed to include *in toto* in this paper, but hopefully the following sketch of its main points will suffice.

If. If ⌜(A → B)⌝ is a theorem of AC, then every conjunct in the basic conjunctive normal form of B is a conjunct in the basic conjunctive normal form of A;
If ⌜(A ↔ B)⌝ is a theorem of AC, then A and B have identical basic conjunctive normal forms.

Proof: 1. First, we must define 'basic conjunctive normal form'. Using the word 'atom' for elementary schemata (i.e., either a negated variable or an unnegated variable) we define first, a *maximal ordered conjunctive normal formula*, abbreviated 'an MOCNF':

Df(MOCNF): A schema, A, is an MOCNF *iff*$_{df}$

(i) Schema A contains only atoms, logical constants '&' and 'v', and parentheses;

(= A is a *normal* form)

(ii) no occurrence of '&' lies in the scope of an occurrence of '∨' in schema A; (= A is a *conjunctive* normal form)

(iii) Schema A is *ordered*; i.e., atoms and larger components are arranged zy a fixed rule of alphabetic and size ordering, are grouped to the right, and there are no redundant conjuncts, or redundant disjuncts in conjuncts.

(iv) Schema A is *maximal*; i.e., if any atom, E_1, occurs anywhere in A but not in some conjunct C_i, then there is a conjunct C_j in A which contains just the atoms in C_i plus E_1.

Next we define 'a *basic conjunctive normal form of A*' abbreviated, 'BCNF(A)':

Df(BCNF): a schema, C, is a BCNF of A iff_{df}
(i) C is an MOCNF and
(ii) ⌜(A ↔ C)⌝ is a theorem of AC.

2. Next, we prove that for every truth-functional schema A, there is a schema C which is a MOCNF and such that ⌜A ↔ C)⌝ is a theorem of AC; i.e., every A has at least one BCNF. To prove this we first have to prove that the following metatheorems and theorem schemata are derivable in AC (we do not include the proofs here):

1. ⌜(A ↔ A)⌝

Rule 2. If ⊢⌜(A ↔ B)⌝ then ⊢⌜(C ↔ C^A//B)⌝ [Where 'C^A//B' represents a schema like C except that an occurrence of A in C is replaced by B]

3.	⌜(--A ↔ A)⌝	Double Negation; cf. AS1
4.	⌜(-(A & B) ↔ (-A ∨ -B))⌝	De Morgan Theorem 1
5.	⌜(-(A ∨ B) ↔ (-A & -B))⌝	De Morgan Theorem 2
6.	⌜(((B & C) ∨ A) ↔ ((A & B) ∨ (A & C))⌝	Distribution 2; AC5 = Distribution 1
7.	⌜(((A & B) & C) ↔ (A & (B & C)))⌝	&-Association 1
8.	⌜(((A ∨ B) ∨ C) ↔ (A ∨ (B ∨ C)))⌝	∨-Association 1
9.	⌜((B & A) ↔ (A & B))⌝	&-Commutation = AC3
10.	⌜((B ∨ A) ↔ (A ∨ B))⌝	∨-Commutation
11.	⌜((A & A) ↔ A)⌝	&-Idempotence 1; cf. AC2
12.	⌜((A ∨ A) ↔ A)⌝	∨-Idempotence 1
13.	⌜((A & B) ↔ (A & (B & (A ∨ B))))⌝	Conjunctive expansion 1
14.	⌜((A & (A ∨ B ∨ C)) ↔ (A & ((A ∨ B) & ((A ∨ C) & (A ∨ (B ∨ C))))))⌝	Conjunctive expansion 2

A procedure is then presented which begins with ⊢⌜(A ↔ B)⌝ for any schema A, and ends with ⊢⌜(A ↔ C)⌝ in which C is MOCNF. This procedure is:

1) Write down ⊢ ⌜(A ↔ A)⌝ [by 1].
2) Derive ⊢ ⌜(A ↔ A_1)⌝ where A_1 is the result of removing abbreviations, except '∨',

from A.

3) Derive $\vdash \ulcorner(A \leftrightarrow A_2)\urcorner$ from 2), where A_2 contains only atoms, logical constants '&' and 'V', and parentheses, using rule 2, and schemata 3, 4 and 5 to bring negation signs in A_1 only next to sentence letters, satisfying (i) of Df(MOCNF).

4) Remove all occurrences of '&' in A_2 from the scope of 'V' to satisfy (ii) of Df(MOCNF), getting $\vdash \ulcorner(A \leftrightarrow A_3)\urcorner$ by using rule 2 with AC5 and 6.

5) Order A_3, getting A_4 so that $\vdash \ulcorner(A \leftrightarrow A_4)\urcorner$ with A_4 satisfying (iii) of Df(MOCNF), by using Rule 2 with 6-12 above to get ordering by size and alphabet, grouping to the right, and elimination of redundancies.

6) Maximize A_4 (re-ordering if necessary) to get C, satisfying (iv) and all other requirements in Df(MOCNF), so that $\vdash\ulcorner(A \leftrightarrow C)\urcorner$, using Rule 2, with 13 and 14.

This procedure provably leads to the desired result. (This result can be gotten in AI and E and indeed in standard logic as well as in AC.)

3. Next we show that for every schema A, there is only one schema C such that C is an MOCNF and $\ulcorner(A \leftrightarrow C)\urcorner$ is a theorem of AC. In other words, in AC every schema A has only one basic conjunctive normal form. (This result is peculiar to AC; it does not hold for standard logic with '≡', for '⊣⊢', or for '↔' as defined by the systems E and AI; this is the most important formal result in AC.) The proof of this point may be sketched as follows:

When a schema is in normal form (satisfying (i) of Df(MOCNF)), all negative occurrences of variables are just the negated sentential variables and all positive occurrences are the unnegated sentential variables. Thus the set of atoms occurring in a normal form schema - i.e., the set of negated sentential variables plus the unnegated sentential variables - is the same as the set of variables which occur positively plus the set of variables which occur negatively in that schema. Since AC satisfies Id, if A and B are both normal forms and $\ulcorner(A \leftrightarrow B)\urcorner$ is a theorem of AC, then A and B have the same set of atoms. Also it can be shown that if $\ulcorner(A \leftrightarrow B)\urcorner$ is derivable in AC, then $\ulcorner(A^* \leftrightarrow B^*)\urcorner$ is derivable in a similar manner, where A^* and B^* are like A and B except that new variables have been uniformly substituted for each variable which has negative occurrences in just its negative occurrences (or alternatively, all negative atoms are uniformly replaced with new variables). Further, it can be shown that if both A and B are MOCNF and A lacks a conjunct B has, or vice versa, then $\ulcorner(A^* \equiv B^*)\urcorner$ can not be a theorem of standard logic. Since we have just seen that if $\ulcorner(A \leftrightarrow B)\urcorner$ is a theorem of AC then $\ulcorner(A^* \leftrightarrow B^*)\urcorner$ is, and we know by Ia, that if $\ulcorner A^* \leftrightarrow B^*)\urcorner$ is a theorem of AC then $\ulcorner(A^* \equiv B^*)\urcorner$ is a theorem of standard logic, it follows that if A lacks a conjunct that B has, or vice versa, then $\ulcorner(A \leftrightarrow B)\urcorner$ is not a theorem of AC. Hence two MOCNFs can be proved to mutually entail each other in AC only if they have all and only the same set of conjuncts; and since they are ordered in the same way, they must be identical. Since '↔' is an equivalence relation - transitive, symmetrical and reflexive - if a schema A is synonymous with two MOCNFs, then they must be synonymous with each other. Thus it follows that every schema A has at most *one* basic conjunctive normal form; i.e., there is *only one* schema (type) C, such that C is a MOCNF and $\ulcorner(A \leftrightarrow C)\urcorner$ is a theorem of AC.

4. It follows from 2 and 3 that condition If holds:

The second part of If, that if ⌜(A ↔ B)⌝ is a theorem of AC, then A and B have the same BCNF, follows quickly from the fact that every schema has one (by step 2) and only one (by step 3) BCNF. For if C is the BCNF(A) and C′ is the BCNF(B), then ⊢⌜ (A ↔ C)⌝ and ⊢⌜ (B ↔ C′)⌝ are both theorems (by definition of BCNF), and by hypothesis ⊢⌜(A ↔ B)⌝; thus, by R2 and R5, it follows that ⊢⌜(C ↔ C′)⌝ which by 2 and 3 is possible only if C and C′ are identical. The first part of If, that if ⌜(A → B)⌝ is a theorem of AC then every conjunct of the BCNF(B) is a conjunct in the BCNF(A), follows from the fact that ⌜(A → B)⌝ is a theorem iff ⌜(A ↔ (A & B))⌝ is by [df '→']; for in the latter case the BCNF of A must contain all and only the same conjuncts as the BCNF of (A & B) and this could not be the case if the BCNF of B contained some conjunct not contained in the BCNF of A, (though BCNF(B) could contain *fewer* conjuncts than BCNF(A)).

Thus both parts of If are satisfied by '→' in AC.

The proof of Ie now follows fairly easily from the proof of If.

Ie. If ⌜(A → B)⌝ is a theorem of AC, then every tautology or inconsistency implicit in B must be implicit in A;
If ⌜(A ↔ B)⌝ is a theorem of AC, then a tautology or inconsistency is implicit in B if and only if it is implicit in A.

Proof: Taking the second part first: By earlier definition, a tautology is *implicit* in a truth-functional schema if and only if it turns up as a tautologous conjunct in the basic conjunctive normal form of the schema. An inconsistency is implicit in a schema if and only if there is a set of 2^n conjuncts of the BCNF of that schema where each conjunct in the set has occurrences of all and only the same n variables (e.g., '...S_1 & -S_1...', '...S_6 & -S_6....', '...(S_1 ∨ S_3) & (S_1 ∨ -S_3) & (-S_1 ∨ S_3) & (-S_1 ∨ -S_3)...', would be such sets). Obviously, by If, if ⌜(A ↔ B)⌝ is a theorem of AC, the BCNF of A is the same as the BCNF of B and thus A and B will have the same implicit tautologies and inconsistencies.

It is equally clear that the first part of Ie will hold in AC:
Since ⌜(A → B)⌝ $=_{df}$ ⌜(A ↔ (A & B))⌝, by arguments given in the proof of If, if ⌜(A → B)⌝ is a theorem of AC, the basic conjunctive normal form of B can contain only such tautologous conjuncts, and such sets of inconsistent conjuncts, as are found in the basic conjunctive normal form of A; which is to say that every tautology and inconsistency implicit in B will be implicit in A.

Thus both parts of Ie hold in AC.

Thus AC satisfies all conditions for tautological entailment as containment of meanings set forth in the criteria Ia through If. The semantic and philosophical import of some of these conditions, particularly If, cannot be pursued here. But such discussion will be enlightened by an investigation of the systems E and AI in the light of the conditions and results in AC so far presented.

III

Anderson and Belnap's system E, and Parry's system AI, are presented for comparison in the tables below. Their full systems obviously include entailments of higher than first degree; each have eight axiom schemata with occurrences of '→' lying within the scope of other occurrences of '→'. Nevertheless, our main points can be made by reference only to the first degree entailments between truth-functional formulae. For, the conflation of containment with deducibility which occurs at this level cannot be eliminated in extensions to higher degree entailments.

Anderson and Belnap's E[a]

^{o}E1. (((A → A) → B) → B)
E2. ((A → B) → ((B → C) → (A → C)))
^{o}E3. ((A → (A → B)) → (A → B))
^{o}E4. ((A & B) → A)
^{o}E5. ((A & B) → B)
^{o}E6. (((A → B) & (A → C)) → (A → (B & C)))
^{o}E7.* ((NA & NB) → N(A & B))
E8. (A → (A ∨ B))
E9. (B → (A ∨ B))
^{o}E10. (((A → C) & (B → C)) → ((A ∨ B) → C))
^{o}E11. ((A & (B ∨ C)) → ((A & B) ∨ C))
^{o}E12. ((A → -A) → -A)
E13. ((A → -B) → (B → -A))
^{o}E14. (--A → A)

[Axioms marked '^{o}' hold in both systems; if not so marked, they fail in the other.]

Rules:
MP. If ⊢X and ⊢(X → Y), then ⊢Y.
ADJ. If ⊢X and ⊢Y, then ⊢(X & Y)
*NX =df ((X → X) → X)

Matrices for consistency proof
Designated values: 1,2,3,4

&	1	2	3	4	5	6	7	8	-	A
1	1	2	3	4	5	6	7	8	8	1
2	2	2	4	4	7	8	7	8	7	2
3	3	4	3	4	6	6	8	8	6	3
4	4	4	4	4	8	8	8	8	5	4
5	5	7	6	8	5	6	7	8	4	5
6	6	8	6	8	6	6	8	8	3	6
7	7	7	8	8	7	8	7	8	2	7
8	8	8	8	8	8	8	8	8	1	8

→	1	2	3	4	5	6	7	8
1	1	8	8	8	8	8	8	8
2	1	2	8	8	7	8	7	8
3	1	8	3	8	6	6	8	8
4	1	2	3	4	5	6	7	8
5	1	8	8	8	4	8	8	8
6	1	8	3	8	3	3	8	8
7	1	2	8	8	2	8	2	8
8	1	1	1	1	1	1	1	1

∨	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	1	2	2
3	1	1	3	3	1	3	1	3
4	1	2	3	4	1	2	3	4
5	1	1	1	1	5	5	5	5
6	1	1	3	3	5	6	5	6
7	1	2	1	2	5	5	7	7
8	1	2	3	4	5	6	7	8

⊃	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	2	1	2	5	5	7	7
3	1	1	3	3	5	6	5	6
4	1	1	1	1	5	5	5	5
5	1	2	3	4	1	3	2	4
6	1	1	3	3	1	3	1	3
7	1	2	1	2	1	1	2	2
8	1	1	1	1	1	1	1	1

a. References: Anderson and Belnap 62, pp.9-24. Axiom set and matrices above are from this article, though matrices are translated so +3=1, +2=2, +1=3, +0=4, -0=5, -2=7, -3=8. Cf. also Anderson and Belnap 75, Ch. IV.

Parry's Analytic Implication[b]

°AI1. ((A & B) → (B & A))
°AI2. (A → (A & A))
°AI3. (A → --A)
°AI4. (--A → A)
°AI5. ((A & (B ∨ C)) → ((A & B) ∨ (A & C)))
AI6. ((A ∨ (B & -B)) → A)
°AI7. (((A → B) & (B → C)) → (A → C))
°AI8. ((A → (B & C)) → (A → B))
°AI9. (((A → B) & (C → D)) → ((A & C) → (B & D)))
°AI10. (((A → B) & (C → D)) → ((A ∨ C) → (B ∨ D)))
°AI11. ((A → B) → (A ⊃ B))
°AI12. (((A ↔ B) & f(A)) → f(B))
AI13. (f(A) → (A → A))
°AI14*. (-(A ⊃ B) → -(A → B))

[Axioms marked '°' hold in both systems; if not so marked, they fail in the other.]

MP. If ⊢ X and ⊢ (X → Y), then ⊢ Y.
ADJ. If ⊢ X and ⊢ Y, then ⊢ (X & Y).
*Added in 1957.

Matrices for consistency proof:
Designated values: 1, 3 (odd numbers)

&	1	2	3	4	-	A
1	1	2	3	4	2	1
2	2	2	4	4	1	2
3	3	4	3	4	4	4
4	4	4	4	4	3	4

→	1	2	3	4
1	1	2	4	4
2	1	1	4	4
3	3	4	3	4
4	3	3	3	3

∨	1	2	3	4
1	1	1	3	3
2	1	2	3	4
3	3	3	3	3
4	3	4	3	4

⊃	1	2	3	4
1	1	2	3	4
2	1	2	3	3
3	3	4	3	4
4	3	3	3	4

b. References: Parry 33. Axiom set, except axiom 14, and matrices taken from this article, though matrices are translated so that 1'=1, 0'=2, 1=3, 0=4. Axiom 14 was added in an unpublished paper in 1957, proved independent by Dunn 72.

That AI and E satisfy condition Ia can be shown in the same way this was shown for AC; by replacing '→' with '⊃' and '↔' by '≡' throughout each system. The matrices accompanying AI and E are not only useful to establish consistency; they can also be used to establish results relating to conditions Ib to Ie. It is immediately obvious by inspection of the axiom schemata of E, that E will not satisfy the criterion of complete variable sharing in Ic; for E2, E8 and E9 all have metavariables in the consequent which are not in the antecedent. But it is provable by the matrices that E will satisfy condition Ib: if ⌜(A → B)⌝ is a theorem of E, then at least one variable in B must occur in A. (Proof: Suppose A and B have no variable in common; then assign 2 or 7 to every variable in A and 3 or 6 to every variable in B. Inspection of the matrices shows that A must take the value 2 or 7 and B must take the value 3 or 6. But the matrix for '→' shows that (2 → 3) = (2 → 6) = (7 → 3) = (7 → 6) = 8; thus in such cases ⌜(A → B)⌝ must take the undesignated value 8 for at least one assignment of values to its variables and thus can not be a theorem of E.)

We can also prove, by Parry's matrices, that Ic holds in AI; if ⌜(A → B)⌝ is a theorem of AI, then every variable which occurs in B will occur in A. (Proof: Assign 1 or 2 to every variable in A and 3 or 4 to any variable in B which does not occur in A. Inspection of all matrices shows that A will take the value 1 or 2 as a whole, while if *any* variable in B has the value 3 or 4, then B will take the value 3 or 4. But the matrix for '→' in AI shows that (1 → 3) = (1 → 4) = (2 → 3) = (2 → 4) = 4. Thus if any variable in B is not contained in A, there will be at least one assignment of values to the variables in ⌜(A → B)⌝ which yields the undesignated value 4 for the entailment. Hence in such cases ⌜(A → B)⌝ can not be a theorem of AI.[5])

Neither system will satisfy Id, hence not Ie or If. For both have the theorem schema

⌜(A → (A ∨ -A))⌝ [E from E8, AI from P13, P11 and Df'⊃'], which violates the condition that the consequent shall not contain a negative occurrence of a variable unless the antecedent does. On the other hand, we shall see that E comes closer to Id and Ie in certain limited respects; for while AI includes ⌜((A ∨ B) → (A ∨ -A))⌝ as a theorem schema [by P13, P11 and Df'⊃'], such schemata do not yield theorems of E [Proof: assign A = 2, B = 1] so that there are some cases where theorems of AI have tautologies in the consequent which are not implicit in the antecedent, but which are not theorems of E. A more general view of this difference will come later. In short we have just shown that E satisfies Ia, and Ib, but not Ic, Id, or Ie or If, while AI satisfies Ia and Ib and Ic, though not Id, Ie or If; on the other hand E comes closer in certain respects to Id, Ie and If than AI does. But to distinguish the different strands of containment and deducibility in these two systems it is helpful to consider some of the motivations involved.

Anderson and Belnap began with a critique of material and strict implication as providing inadequate accounts of valid inference. Starting with a Fitch-style account of natural deduction, which relies heavily on conditional proof, they devised a set of rules for subscripting entries in natural deduction schemata so as to keep track of whether a given formula was *used* or not in getting from a given assumption to a conclusion. They held that ⌜A entails B⌝ if and only if there is a valid inference from A to B, that there can be no valid inference from A to B unless A is *relevant* to B, and A can not be relevant to B unless it can be *used* in the inference from A to B. When these rules were converted into entailment schemata, it turned out that for A to be relevant to B, A must contain at least one variable occurring in B. But Fitch-style deduction rules, even when restricted by subscripts, do not involve any clear concept that the consequent or conclusion must be *contained in* the premisses. They assume not only that A ⊢ (A ∨ B), but also that ((A → B) ⊢ ((B → C) → (A → C)) and a great many other deductions, are valid in which various components of the conclusion do not occur at all in the premisses. It seems hard to deny that in such cases if the premisses were *true* the conclusion would have to be *true* also, i.e., it is hard to deny some connection between these rules and valid deduction. But may not one be working at cross-purposes if one tries to associate "entails" with both containment of meanings and deducibility at the same time? Anderson and Belnap have tried to do both; and the result has been that they have not completely succeeded at either. Calling ⌜(A → B)⌝ a "primitive entailment" if A is a conjunction of atoms and B is a disjunction of atoms, they say that a primitive entailment is "explicitly tautological" if some conjoined atom of A is identical with some disjoined atom of B and add "Such entailments may be thought of as satisfying the classical dogma that for A to entail B, B must be "contained" in A" [cf. Anderson and Belnap 75, pp.154-5]. Then they show that in E a first degree entailment ⌜(A → B)⌝ is a theorem if and only if the disjunctive normal form of A, ⌜(A_1 ∨...∨ A_n)⌝, and the conjunctive normal form of B, ⌜(B_1 &...& B_m)⌝, are such that each ⌜(A_i → B_j)⌝ is an explicitly tautological entailment. Thus we have a syntactical condition for first degree entailment in E which may be compared with the condition If (first part) which says that every conjunct of the conjunctive normal form of B must be a conjunct of the conjunctive normal form of A. The difference is explained by the fact that Anderson and Belnap wish to include Addition as a principle of entailment; but they do not settle doubts about their claim that, e.g., '(S_4 & S_3)' *contains* ('S_1 ∨ -S_6 ∨ S_3)' or the question why Addition should be considered as satisfying the classical concept of containment. One can only conjecture that they have confusedly supposed that if it is true that if A were true then B would have to be true, then B must be "contained" in A; but in what sense of "contained"? Thus it seems that in a fairly straightforward sense of "contains", E fails to give a clear concept of entailment of meanings. But strangely this same imperfect effort to capture containment in "tautological entailment" becomes the ground in E for rejecting certain widely accepted patterns for valid deductions [Anderson and Belnap 75, p.164];

(-A ∨ B)	-(A & B)	(A ⊃ B)	(A ⊃ B)
A	A	A	(B ⊃ C)
--------	---------	--------	-------------
Hence, B	Hence, -B	Hence, B	Hence, (A ⊃ C)

which have played central roles in ancient and/or modern classical logic. The corresponding principles, ⌜((A & (-A ∨ B)) → B)⌝, ⌜((A & -(A & B)) → -B)⌝, ⌜((A & (A ⊃ B)) → B)⌝ and ⌜(((A ⊃ B) & (B ⊃ C)) → (A ⊃ C))⌝ are not theorems of E, as can be seen either by assigning A = 2, B = 3, C = 4, or by reducing the antecedents to disjunctive normal form and the consequents to conjunctive normal form and applying the syntactical test given above. In addition, by the latter test we find that the following are not theorems of E:

⌜((A ∨ B) → ((A & B) ∨ (A & -B) ∨ (-A & B)))⌝
⌜((A ⊃ B) → ((A & B) ∨ (-A & B) ∨ (-A & -B)))⌝
⌜((A ≡ B) → ((A & B) ∨ (-A & -B)))⌝

If we assume that all instances of sentence schemata must obey the law of excluded middle, then from the *truth* of the antecedents and this assumption the truth of the consequents must surely follow. Thus, despite Anderson and Belnap's ingenious argument, the sense that E omits valid patterns of deduction persists. The omission of these principles would be of no consequence save for the fact that Anderson and Belnap purport to formalize entailment as the converse of 'is deducible from'. If that is their intent they seem clearly to be missing something here. But the interesting thing is that their "independent proof" that these are not valid forms of inference is based on their imperfect and partial treatment of entailment as containment, i.e., in the syntactical test above; e.g., ⌜((A & (-A ∨ B)) → B)⌝ does not hold because ⌜((A & -A) ∨ (A & B)) → B)⌝ is not a tautological entailment since the disjunct ⌜(A & -A)⌝ does not contain B. In holding that the consequent, in all of these cases, is not *contained* in the antecedent they are, by our conditions Ia-If, entirely correct. Further, in these cases they are more correct than Parry's system AI (which includes all of these omitted schemata as theorems) if entailment is to be treated as containment. For Parry violates the principle which refuses to say that B is contained (in the sense of entailment) in (A ∨ B) in a certain sense, by allowing B to be entailed by ⌜(B ∨ (A & -A))⌝ [P6], and ⌜(A ∨ -A)⌝ to be entailed by ⌜(A ∨ B)⌝ [by P13, P11 and Df'⊃'].

Turning then to Parry's system AI, we find a different conflict between concepts of entailment as deducibility and entailment as containment. Parry was also struck by the inadequacy of Lewis's attempt to capture the concept of deducibility through strict implication. And like Anderson and Belnap (only twenty-five to thirty years earlier) he held that in some sense what is in the conclusion must be contained in the premisses. In his first published work on the subject he connected the fact that ⌜(A → B)⌝ was a theorem of AI only if all variables in B occurred in A, with the concept of logical consequence by which the conclusion could not contain any concepts not contained in the premisses. This concept of logical consequence is clearly different from a concept of deducibility based on the requirement that the *truth* of the conclusion be deducible from the *truth* of the premisses; for the former excludes immediately the principle of Addition, and various principles involving nested conditionals in the consequent, such as ⌜((A → B) → ((B → C) → (A → C)))⌝ and others which are treated as valid principles of inference in Fitch-style theories of deduction, as in Anderson and Belnap. But regardless of what concept of logical consequence Parry had in mind, the relation of his system to a notion of logical implication or entailment in the sense of containment was clear and strong. It is immediately clear by inspection of the formulae why AI includes the theorem schemata in list I below but excludes the schemata in List II from theoremhood:

I. ((A & B) → A)
((A & B) → B)
((A & (A → B)) → B)
(((A → B) & (B → C)) → (A → C))
((A → (B → C)) → ((A → B) → (A → C)))

II. (A → (A ∨ B))
(A → (B → B))
(A → ((A → B) → B))
((A → B) → ((B → C) → (A → C)))
((A → B) → ((A → (B → C)) → (A → C)))

In each case it is clear that the consequents of entailments in list II all contain variables which do not occur in the antecedents whereas this is not the case for schemata in list I. Further it is clear why transposition, ⌜((A → B) → (-B → -A))⌝, and exportation, ⌜(((A & B) → C) → (A → (B → C)))⌝, are not admissible in AI; both of these would convert the first four schemata in list I into schemata with variables in the consequent which were not in the antecedent. Similarly, permutation, ⌜((A → (B → C)) → (B → (A → C)))⌝ is inadmissible because it would convert the fifth schema in list I into the fifth schema in list II or the obvious example of containment ⌜((A → B) → (A → B))⌝ into the counterexample ⌜(A → ((A → B) → B))⌝. On the other hand importation, ⌜((A → (B → C)) → ((A & B) → C))⌝, and modus ponens, ⌜((A & (A → B)) → B)⌝, are theorems of AI and provably free from such deviations from containment. These relatively simple and straightforward explanations of inclusions and omissions contrast with the much more complicated and often less clear explanations offered for inclusions or omissions from E. Why for example, should we admit ⌜((A → B) → ((B → C) → (A → C)))⌝ as capturing entailment in the sense of the converse of "is deducible from", but reject ⌜(A → ((A → B) → B))⌝ in E? Anderson and Belnap's answer claims that the latter commits a fallacy of modality, inferring a necessary proposition from a contingent one; but why is not the same objection raised against their principle of syllogism? The responses, to say the least, are extremely subtle.

Yet Parry does not conceive of his system solely as a system of entailment in the sense of containment. In various places he treats it as a candidate for the converse of "is deducible from". As such his elimination of such principles as Addition, $\ulcorner(A \rightarrow (A \vee B))\urcorner$, the factor $\ulcorner((A \rightarrow B) \rightarrow ((A \,\&\, C) \rightarrow (B \,\&\, C))\urcorner$ or even $\ulcorner((A \leftrightarrow B) \rightarrow ((A \,\&\, C) \leftrightarrow (B \,\&\, C)))\urcorner$, not to mention principles with nested conditionals like the last three in list II above, runs counter to that notion of valid inference related to determinations of whether the conclusions would have to be true if - or in the event that - the premisses were true. Thus Anderson and Belnap and others might rightly dispute whether AI has captured precisely the concept of deducibility. But unfortunately AI fails also to capture a completely clear notion of entailment as containment of meanings. Several gaps in deducibility which we found in E are filled in AI, but in filling them AI forfeits the strict concept of entailment as containment. Thus $\ulcorner((A \,\&\, (\text{-}A \vee B)) \rightarrow B)\urcorner$, $\ulcorner((A \,\&\, (A \supset B)) \rightarrow B)\urcorner$, $\ulcorner(((A \supset B) \,\&\, (B \supset C)) \rightarrow (A \supset C))\urcorner$, $\ulcorner((A \vee B) \rightarrow ((A \,\&\, B) \vee (A \,\&\, \text{-}B) \vee (\text{-}A \,\&\, B)))\urcorner$ and $\ulcorner((A \equiv B) \rightarrow ((A \,\&\, B) \vee (\text{-}A \,\&\, \text{-}B)))\urcorner$ are all theorems of AI. More broadly, in AI it can be proved that every schema mutually entails (or analytically implies) its "full disjunctive normal form". This is no small consequence. The full disjunctive normal form can be formed directly from the standard truth-table of a schema A as follows: construct a disjunction such that each row in the truth-table of A in which A as a whole takes the value T is represented by just one disjunct, and this disjunct is a conjunction of atoms such that each sentence letter in A which takes F in that row occurs negated in the conjunction and each sentence letter which takes T in that row occurs unnegated in the conjunction. Thus, for example, the full disjunctive normal form of $\ulcorner(A \vee B)\urcorner$ is just $\ulcorner((A \,\&\, B) \vee (\text{-}A \,\&\, B) \vee (A \,\&\, \text{-}B))\urcorner$. Every consistent truth-functional scheme can be proven in AI to mutually entail (or be "analytically equivalent" to) a normal form which uniquely represents its own truth-table! But however desirable this result may be from the point of view of deducibility, such results are not tenable if entailment is taken as involving containment in the strict and straightforward sense we have advanced (or in the weaker sense of Anderson and Belnap either). Obviously, the full disjunctive normal form of $\ulcorner(A \vee B)\urcorner$ contains negative occurrences of letters which do not occur negatively in $\ulcorner(A \vee B)\urcorner$. This same result will hold for many other schemata, for to reduce all consistent schemata to "*full* disjunctive normal form" we need more than the principles of Double Negation, De Morgan Laws, Distribution, Association, Commutation and Idempotence available in AC and in E. We need also such principles as $\ulcorner((A \vee B) \rightarrow (A \vee \text{-}A))\urcorner$ and $\ulcorner((A \vee (B \,\&\, \text{-}B)) \rightarrow A)\urcorner$, gotten by P13 and P6 in AI, by which we can drop inconsistent disjuncts and see that every sentence letters occurs either negated or not in each disjunct. These principles are not available in E or AC, nor should they be if entailment is taken as containment of meanings. For, as we mentioned earlier, we do not want to say that $\ulcorner(A \vee B)\urcorner$ entails A merely because A occurs in $\ulcorner(A \vee B)\urcorner$. But why, then, should we want to say that $\ulcorner(A \vee (B \,\&\, \text{-}B))\urcorner$ entails, or contains the meaning of, A? Any impetus to do so is not on the grounds of containment of meaning in the sense required. Rather, mostly likely, it is on the grounds that since we know, by the law of non-contradiction, that $\ulcorner(B \,\&\, \text{-}B)\urcorner$ can not be true, we must conclude that if ⌜(A ∨ (B & -B))⌝] were *true*, then A would have to be *true*. But this argument concerns deducing the *truth* of A from the *truth* of ⌜(A ∨ (B & -B))⌝, not containment of A's meaning, in the relevant sense, in ⌜(A ∨ (B & -B))⌝. Anderson and Belnap are right in saying that ⌜(A ∨ (B & -B))⌝ does not contain the meaning of A since A has no occurrence in one of the disjuncts. But they are wrong in omitting the fact that the truth of A is deducible from that of ⌜(A ∨ (B & -B))⌝. Their rejection of ⌜((A & (-A ∨ B)) → B)⌝ is also right and wrong in the same two respects. B is not contained in either of the conjuncts of the antecedent (or in both disjuncts of the equivalent schema ⌜(((A & -A) ∨ (A & B)) → B)⌝; but assuming only the law of non-contradiction, B's truth would surely be deducible from the truth of the antecedent, despite their disclaimers.

IV

We have argued that both Parry's system AI and Anderson and Belnap's system E include too much for a theory of logical containment in the strict and plausible sense we have advanced; though both systems move significantly in this direction away from standard logic. On the other hand, we have argued that neither has presented an adequate formalization of deducibility, though both have theorems which seem clearly related to 'is deducible from' and go beyond our criteria for containment. Obviously, our position implies that logical containment is a stricter concept than deducibility; we want to agree that if A contains, or is synonymous with B, then B is deducible from A. But we do not want the converse, that whenever B is deducible from A, the full referential meaning of B is contained in the meaning of A.

What plausible suggestions, then might be made with respect to an appropriate theory of deducibility?

It might be thought, in the light of the preceding discussion, that analytic containment in AC is just the intersection of the systems of first degree entailments between truth-functional schemata in AI and E, and that perhaps, since both AI and E included some plausible claims for deducibility theorems beyond those covered by containment, that deducibility plus containment might be captured by the union of AI and E. But this is wrong on both counts. AC is even stronger than the intersection of the first-degree entailment fragments of AI and E, for both of the latter have ⌜(A → (A ∨ -A))⌝ as a theorem, while AC does not have this, since it violates conditions Id, Ie and If. (In E ⌜(A → (-A ∨ A))⌝ is a substitution instance of E9; in AI it is gotten from AI13, AI11 and df'⊃'). Thus AC is not in the intersection of E and AI. And the union of E and AI yields the very paradoxes of strict implication which all three systems unite in rejecting as inappropriate deducibility principles. For by E9 we have ⊢ ⌜((A & -A) → (B ∨ (A & -A))⌝, by AI6 we have ⊢ ⌜(B ∨ (A & -A)) → B)⌝ and thus by hypothetical syllogism, which holds in both systems, we get ⊢ ⌜((A & -A) → B)⌝. Thus the distinction between containment and deducibility cannot be defined by inter-relationships of AI and E, nor can appropriate formalizations of each of these concepts be secured by this method. What other approaches might be suggested?

At various points we have suggested that though a given wff, A, may not *contain* (in our sense) a wff, B, nevertheless the *truth of* B might be *deducible from* the *truth* of A. Thus while we deny (vs. Anderson and Belnap) that A *contains* $\ulcorner(A \vee B)\urcorner$, we admit that on the truth-functional interpretation of '$\vee$', the *truth of* A is an analytically sufficient condition for asserting the *truth of* $\ulcorner(A \vee B)\urcorner$. Again, while we deny (vs. Parry) that $\ulcorner(A \vee (B \,\&\, \text{-}B))\urcorner$ logically contains A, we agree that from the *truth of* $\ulcorner(A \vee \text{-}B \,\&\, \text{-}B))\urcorner$ we could logically deduce the *truth of* A. One suggestion that seems worthy of study, then, is that the distinction between containment and deducibility can be established by the introduction of a truth-operator, 'T', such that $\ulcorner TA\urcorner$ is read $\ulcorner$ It is true that $A\urcorner$(just as $\ulcorner\text{-}A\urcorner$ is sometimes read $\ulcorner$it is false that $A\urcorner$). By this device we can express various principles which go beyond containment, e.g., $\ulcorner(TA \rightarrow T(A \vee B))\urcorner$] for $\ulcorner$If it is true that A then it is true that either A or $B\urcorner$, and $\ulcorner(T(A \vee (B \,\&\, \text{-}B)) \rightarrow TA)\urcorner$ for $\ulcorner$If it is true that either A or both B and not B, then it is true that $A\urcorner$. Such a theory, with truth-operators, could be called truth-theory and should be included in the corpus of formal logic. Provided the theory has a rule of Modus Ponens for the conditional represented by '$\rightarrow$', the principles above would then immediately yield derived deduction rules such as, $TA \vdash T(A \vee B)$. Although we can not present a completely satisfactory formal system along these lines at this time, we will provide a four-value matrix set which establishes the consistency of a very close approximation, and thus, we hope, adds credibility to the project.

Before proceeding further, we must revise somewhat our interpretations of the notation we have been using. For convenience we have associated the arrow, '$\rightarrow$', up to this point with the concept of containment, and '$\leftrightarrow$' with that of synonymity. But now we shall treat '$\rightarrow$' and '$\leftrightarrow$' as symbols solely for conditionals and biconditionals; $\ulcorner(A \rightarrow B)\urcorner$ is read $\ulcorner$(If A then $B)\urcorner$. Containment and synonymity (as mentioned earlier) are strictly speaking metalinguistic concepts, better expressed formally in $\ulcorner$('A' contains 'B')$\urcorner$; e.g., "(Jo died and Flo wept)' contains 'Flo wept''. All of the first degree theorems which we have presented so far may now be viewed as schemata of biconditionals which are logically true *by virtue of* mutual containment or containment of the consequent in the antecedent. In place of the definition, $\ulcorner(A \rightarrow B)\urcorner =df \ulcorner(A \leftrightarrow (A \,\&\, B))\urcorner$, we have $\ulcorner$('A' contains 'B')$\urcorner$ =df $\ulcorner$('A' is synonymous with '(A & B)')$\urcorner$. A stricter presentation would develop first a formal theory of containment and synonymity in the metalanguage, then link it with conditionals by some deduction rules such as, from $\ulcorner$('A' is synonymous with 'B')$\urcorner$ deduce $\ulcorner(A \leftrightarrow B)\urcorner$] is logically true, or, from $\ulcorner$('A' contains 'B')$\urcorner$ deduce $\ulcorner(A \rightarrow B)\urcorner$, or, from $\ulcorner$('A' contains 'B')$\urcorner$ deduce $\ulcorner(TA \rightarrow TB)\urcorner$.

Our objective here is to separate the sheep from the goats, or more accurately, the conditionals based on containments from those based on truth-theory, while allowing that both groups are composed of logically true conditionals from which deduction rules will follow. Thus we count $\ulcorner(TA \rightarrow T(A \vee B))\urcorner$ as a logically true conditional, which yields the deduction rule $TA \vdash T(A \vee B)$, not because $\ulcorner TA\urcorner$ contains $\ulcorner T(A \vee B)\urcorner$, but because from the truth-functional meaning of '$\vee$' it is clear analytically that if A is true then $\ulcorner(A \vee B)\urcorner$] ust be counted as true also. On this account what have traditionally been treated as semantic rules for standard truth-functional connectives will be incorporated into the corpus of propositional logic by means of the truth-operator, while kept distinct from containment. The payoff is not only that we achieve a theory of synonymity and containment which is unattainable in the truth-functional logic, but that we eliminate "paradoxes" of material and strict implication in the process.

The conditional represented by '$\rightarrow$' in this approach can not be the truth-functional conditional represented here by '$\supset$'. This is because we reject (as do Parry, Anderson and Belnap) the account of deducibility which goes along with standard truth-functional logic and its truth-functional conditional although we accept, with all logicians (including Parry, Anderson and Belnap), the following principle:

A. $\ulcorner$(If A then B)$\urcorner$ is logically true if and only if B is logically deducible from A [or, ($\vdash \ulcorner(A \rightarrow B)\urcorner \leftrightarrow A \vdash B$))].[6]

We do not deny that $\ulcorner((A \,\&\, \text{-}A) \supset B)\urcorner$ is logically true - indeed it will be a theorem of logic, because, on removing abbreviations it amounts to simply a denial of an inconsistency. What we deny is that B is logically *deducible* from $\ulcorner(A \,\&\, \text{-}A)\urcorner$ or from every inconsistency, or, that every logical truth is *deducible* from any statement whatever, or, that $\ulcorner$(If A then B)$\urcorner$ is *deducible* from B or from $\ulcorner$-A$\urcorner$, and so on. But all of these consequences, which we are pledged to avoid, would follow if we accepted the principle A above and also accepted the truth-functional conditional as an interpretation of '$\rightarrow$'.

We do not now have, nor do we need to have, a complete account of what conditional must be put in the place of the truth-functional conditional. What we do have, and all that we need for present purposes, is a set of necessary conditions which in addition to A above, must be met by such conditionals. These conditions (which will leave all and only the present *theorems* of standard logic intact as the logical truths of O-degree wffs), are listed in B to G below.

B. The rule of *Modus Ponens* should hold. Thus the following should be laws of logic:

$$(TA \rightarrow (T(A \rightarrow B) \rightarrow TB))$$
$$(T(A \,\&\, (A \rightarrow B)) \rightarrow TB)$$
$$((TA \,\&\, T(A \rightarrow B)) \rightarrow TB)$$

C. The truth of the truth-functional conditional should follow from the truth of a genuine conditional - though the converse does not hold and the former does not contain the latter. Thus the following laws of logic should obtain, as all parties will agree:

$$(T(A \rightarrow B) \rightarrow T(A \supset B))$$
$$(T(A \rightarrow B) \rightarrow T(\text{-}A \vee B))$$
$$(T(A \rightarrow B) \rightarrow T\text{-}(A \,\&\, \text{-}B))$$

But the following should *not* be laws of logic:

$(T(A \supset B) \rightarrow T(A \rightarrow B))$
$((A \supset B) \rightarrow (A \rightarrow B))$

D. The set of logically true conditionals must *not* include "paradoxes" of strict or material implication. In this we, along with Parry, Anderson and Belnap, diverge from standard logic. Thus the following must *not* be logical truths:

$((A \,\&\, \text{-}A) \rightarrow B)$
$(B \rightarrow (\text{-}A \vee A))$
$(B \rightarrow (A \rightarrow B))$
$(\text{-}A \rightarrow (A \rightarrow B))$

(though '⊃' - for '→' 0-degree analogues of these will be theorems). But also we must *not* allow as logical theorems:

$(TB \rightarrow T(\text{-}A \vee A))$
$(T(A \,\&\, \text{-}A) \rightarrow TB)$
$(\text{-}T(\text{-}A \vee A) \rightarrow TB)$
$(TB \rightarrow T(A \rightarrow B))$
$(T\text{-}A \rightarrow T(A \rightarrow B))$

E. The following principles, without T-operators, which are axioms or theorems of Anderson and Belnaps' E, but are excluded from Parry's system, should *not* be logical theorems as they stand since (having variables in the consequent not present in the antecedent) they can not be established on containment alone:

$((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))$ [E2]
$(A \rightarrow (A \vee B))$ [E8]
$(B \rightarrow (A \vee B))$ [E9]
$(\text{-}A \rightarrow (A \supset B))$
$(B \rightarrow (A \supset B))$

On the other hand, the following principles, which seem to satisfy the intuitions appealed to in support of principles just excluded, *should* be theorems of logic:

$(T(A \rightarrow B) \rightarrow (T(B \rightarrow C) \rightarrow T(A \rightarrow C)))$
$(TA \rightarrow T(A \vee B))$
$(TB \rightarrow T(A \vee B))$
$(T\text{-}A \rightarrow T(A \supset B))$
$(TB \rightarrow T(A \supset B))$

the logical truth of these latter being due to truth-theory, not containment.

F. The following principles without T-operators, which are axioms or theorems of Parry's system, but are excluded from Anderson and Belnap's system E, should *not* be theorems of logic, for various reasons referred to in preceding sections:

$((A \vee (B \,\&\, \text{-}B)) \rightarrow A)$ [AI6]

$(f(A) \rightarrow (A \rightarrow A))$ [AI13]

$(A \rightarrow (A \rightarrow A))$

$((A \,\&\, B) \leftrightarrow ((A \vee B) \,\&\, ((A \vee -B) \,\&\, (-A \vee B))))$

$((A \,\&\, (-A \vee B)) \rightarrow B)$

$((A \,\&\, (A \supset B)) \rightarrow B)$

On the other hand, the following principles seem to satisfy the intuitions appealed to in support of all, except the second, of these excluded principles, and thus *should* be theorems of logic:

$(T(A \vee (B \,\&\, -B)) \rightarrow TA)$

$(TA \rightarrow T(A \rightarrow A))$

$(T(A \,\&\, B) \leftrightarrow (T\,(A \vee B) \,\&\, (T(A \vee -B) \,\&\, T(-A \vee B))))$

$(T(A \,\&\, (-A \vee B)) \rightarrow TB)$

$((TA \,\&\, T(-A \vee B)) \rightarrow TB)$

$((TA \,\&\, T(A \supset B)) \rightarrow TB)$

The last two of course, yield the disjunctive syllogism and the standard rule of detachment (often called 'modus ponens' in standard logic) in truth theory versions of deduction rules. But this in no way allows that from the truth of $T(A \supset B)$ one can get the rule $TA \vdash TB$; e.g., though $\ulcorner(A \supset (B \supset B))\urcorner$ and $\ulcorner T(A \supset (B \supset B))\urcorner$ may be theorems, it does not follow that $TA \vdash T(B \supset B)$, $A \vdash (B \supset B)$, will be derivable as deduction rules.

G. By E and F we have eliminated principles of both Anderson and Belnap's system and Parry's system, which stand in the way of a theory of synonymity and containment, while allowing suitable replacements for excluded theorems by means of the truth-operator. But we still want to reject from our system the following, for reasons explained earlier, though they are theorems in one, or both of these other systems:

$(A \rightarrow (-A \vee A))$

$((A \,\&\, (B \,\&\, -B)) \rightarrow (B \,\&\, (A \,\&\, -A)))$

$(((A \,\&\, -A) \,\&\, (B \vee -B)) \rightarrow ((A \vee -A) \,\&\, (B \,\&\, -B)))$

These are cases which violate the concept of containment by allowing a contradictory or tautologous statement about a subject in the consequent of a conditional though it was not contained in the antecedent.

H. Finally, we shall want all and only the standard truth-functional tautologies to be theorems where there are no 'T's or '→'s or '↔'s in the wffs. And we shall want all of the axioms AC1 through AC5 and AC's rule R1 [cf. above] to obtain.

With two exceptions mentioned below, the following set of matrices establishes the consistency of any system which meets all the conditions, positive and negative, listed in A through G above:

Designated values: 1,2

TA		-A	
1	1	4	1
2	2	3	2
4	3	2	3
4	4	1	4

(A & B)	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

(A → B)	1	2	3	4
1	1	4	3	4
2	1	2	4	3
3	1	4	2	4
4	1	1	1	1

'$\vee$', '$\supset$', and '$\equiv$' are defined in the usual fashion from '&' and '-'.

$\ulcorner(A \leftrightarrow B)\urcorner$ =df $\ulcorner((A \rightarrow B) \,\&\, (B \rightarrow A))\urcorner$.

This matrix set will also satisfy what have traditionally been treated as semantic rules for truth-functional connectives, especially if we define 'FA' for 'it is false that A' (vs 'it is not the case that A' for '-A') as $\ulcorner FA\urcorner$ =df $\ulcorner T\text{-}A\urcorner$:

$(T(A \,\&\, B) \rightarrow (TA \,\&\, TB))$
$((TA \,\&\, TB) \rightarrow T(A \,\&\, B))$ [From which a rule of adjunction can be derived]
$(T(A \vee B) \leftrightarrow (TA \vee TB))$
$(F(A \vee B) \leftrightarrow (FA \,\&\, FB))$
$(T\text{-}A \leftrightarrow FA)$
$(TA \leftrightarrow F\text{-}A)$ etc.

Although the matrix set rejects the implication fragment of E as axiomatized by E1, E2 and E3 with Modus Ponens, it does include Modus Ponens and the following truth-theory analogues of E1, E2 and E3:

$(T((A \rightarrow A) \rightarrow B) \rightarrow TB)$
$(T(A \rightarrow B) \rightarrow (T(B \rightarrow C) \rightarrow T(A \rightarrow C)))$
$(T(A \rightarrow (A \rightarrow B)) \rightarrow T(A \rightarrow B))$

A formula is tautologous according to this matrix set if and only if it takes only 1's and 2's in its truth-table. All of the wffs which have been proposed for inclusion among logical truths above are tautologous and all of those scheduled for exclusion are non-tautologous, with the following two exceptions: 1) in place of $\ulcorner((A \,\&\, B) \rightarrow B)\urcorner$ we must make do with $\ulcorner(T(A \,\&\, B) \rightarrow TA)\urcorner$ since the former is not a tautology on this model, and 2) although we exclude $\ulcorner((A \,\&\, \text{-}A) \rightarrow B)\urcorner$, $\ulcorner(TB \rightarrow T(\text{-}A \vee A))\urcorner$ and $\ulcorner(\text{-}T(\text{-}A \vee A) \rightarrow TB)\urcorner$ from tautologies in this model, the unwanted $\ulcorner(T(A \,\&\, \text{-}A) \rightarrow TB)\urcorner$ comes out a tautology. Conceivably one or both of these difficulties could be accounted for or removed either by finding a better model, or by some fine tuned revisions in the semantic theory underlying our judgments above. But we are not here proposing any complete or final theory. What we have presented has not been an axiomatized theory, much less a formal semantic theory, and even less a proof of the completeness of some formal theory with respect to a plausible formal semantics. Nevertheless, we hope that our main point has been accomplished, namely that of establishing the credibility of the possibility of a theory of logic which eliminates the "paradoxes" of strict and material deducibility, permits a rigorous and viable theory of synonymy and containment, incorporates the semantics of truth-functional connectives in logic, and preserves all the theorems of classical logic while excluding the classical non-theorems - in short preserves the good, eliminates the bad, and adds improvements to the classical theory of logic.

FOOTNOTES

1. In particular, Anderson and Belnap, in 75, speak of entailment as the "converse of deducibility", so that "$(A \rightarrow B)$" will be interpreted as "A entails B" or "B is deducible from A". Cf. pp.5,7.

2. Anderson and Belnap, 75. The system, E, of entailment is formulated axiomatically on pages pp.231-232. However, we shall be dealing in this paper only with the fragment *fde*, which contains only those theorems which are first-degree wffs, i.e., have no occurrences of '$\rightarrow$' within the scope of another '$\rightarrow$'. An axiomatization of fde is given in §15.2. On "containment" cf. p.155.

3. In Jacques Herbrand 30, cf. paraphrase in van Heijenoort 67, p.528.

4. We would have preferred to use ⌜('A' contains 'B')⌝ and ⌜('A' is synonymous with 'B')⌝ instead of ⌜$(A \rightarrow B)$⌝ and ⌜$(A \leftrightarrow B)$⌝ in AC. But convenience and precedent argue against this level of metalanguage. Preference and convenience can be reconciled by supposing that in AC ⌜$(A \rightarrow B)$⌝ abbreviates ⌜($(A \supset B)$ & 'A' contains 'B')⌝ and that ⌜$(A \leftrightarrow B)$⌝ abbreviates ⌜($(A \leftrightarrow B)$& 'A' is synonymous with 'B')⌝.

5. This same proof may be used to show that AC satisfies $I\underline{c}$; for the matrices given for AI also serve as a consistency model for AC.

6. This principle is deducible in standard truth-functional metalogic from the rule of detachment (called 'modus ponens') and the Deduction Theorem. But, as Anderson and Belnap have correctly pointed out, the Deduction Theorem itself allows much too much, including the paradoxes of strict and material deducibility which they, and I, are pledged to eliminate. Cf. Anderson and Belnap 75, §22.2.1.